Inspired Inequalities

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Let $x_1, \ldots, x_n$ be nonnegative real numbers. Define the elementary symmetric polynomials $s_1, \ldots, s_n$ by $(x + x_1) \cdots (x + x_n) = x^n + \cdots + s_{n-1}x + s_n$. The symmetric averages d_i are then defined by $d_i = s_i / \binom{n}{i}$.

Trick 1 (Symmetric Polynomials) *If $x_1, \ldots, x_m$ are real numbers and $m > n$, then there exist reals $x'_1, \ldots, x'_{m-1}$ such that $d'_i = d_i$ for $i = 1, \ldots, n$. In particular, an inequality in symmetric averages $d_1, \ldots, d_n$ need only be proved for the n-variable case.*

Theorem 1 (Newton) $d_i^2 \geq d_{i+1}d_{i-1}$.

Theorem 2 (Maclaurin) $d_1 \geq \sqrt{d_2} \geq \sqrt[3]{d_3} \geq \cdots \geq \sqrt[n]{d_n}$. b

Theorem 3 (Bernoulli) *For all $r \geq 1$ and $x \geq -1$,*

$$(1 + x)^r \geq 1 + xr$$

A variety of substitutions can be useful when solving inequalities. There is no special prize for using the given variables. Substitutions can be a particularly handy way to deal with constraints; if a constraint is unwieldy, change it!

- Positive reals $x_1, \ldots, x_n$ such that the product $x_1 \cdots x_n = 1$ can be written as $x_i = y_i/y_{i-1}$ or $x_i = y_{i-1}/y_i$ for unconstrained positive reals $y_1, \ldots, y_n$.
- Positive reals $x_1, \ldots, x_n$ such that the product $x_1 \cdots x_n = 1$ can be written as $x_i = e^{y_i}$ for reals $y_1, \ldots, y_n$ with sum 0.
- Three positive reals a, b, c which are the sides of a triangle can be written $a = x+y, b = y+z, c = z+x$ for unconstrained positive reals x, y, z.
- Three positive reals a, b, c such that $a + b + c = abc$ are expressible as $a = \tan A, b = \tan B, c = \tan C$ where ABC is a triangle. (What is the related cotangent substitution?)
- Three positive reals a, b, c such that $a^2 + b^2 + c^2 + 2abc = 1$ are expressible as $a = \cos(A), b = \cos B, c = \cos C$ where ABC is an acute triangle.

1 Isolated Fudging

Some contest inequalities are solved by blending all of the terms under consideration together with one of the standard inequalities. That strategy sometimes fails, in which case the alternative is manipulating each term by itself (hence the name) so that the resultant sum is simple.

One idea is to give a linear approximation, which can be accomplished in several ways. We can compare given terms to

$$\frac{k \cdot a^r}{a^r + b^r + c^r},$$

which gives a sum independent of r. (The free variable r is calculated by taking derivatives at an equality case, and actually can be at most one value, while k is the desired sum.) Equivalently, we can fix $a+b+c$ and attempt to give a linear approximation. In the following solution, $r=1$ can be found by computing

$$\frac{\partial}{\partial a}\left(\frac{2a}{b+c}\right)^{2/3}\Bigg|_{(a,b,c)=(1,1,1)} = \left(\frac{2}{b+c}\right)^{2/3}\cdot(2/3)a^{-1/3}\Bigg|_{(a,b,c)=(1,1,1)} = 2/3$$

and equating it with

$$\frac{\partial}{\partial a}\frac{3a^r}{a^r+b^r+c^r} = \frac{2r}{3}$$

It is then an easy check to see that the choice $r=1$ is suitable. Importantly, a complete solution need not, and should not, show the work behind finding r.

(MOP 2002) a,b,c are positive reals. Show that

$$\left(\frac{2a}{b+c}\right)^{2/3}+\left(\frac{2b}{c+a}\right)^{2/3}+\left(\frac{2c}{a+b}\right)^{2/3}\geq 3$$

Solution. Consider the inequality

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}}\geq\frac{3a}{a+b+c}$$

This is equivalent to

$$a+b+c\geq 3\sqrt[3]{a\left(\frac{b+c}{2}\right)^2}$$

and is a direct consequence of AM-GM on a, $\frac{b+c}{2}$, and $\frac{b+c}{2}$. It follows that

$$\sum_{cyc}\left(\frac{2a}{b+c}\right)^{\frac{2}{3}}\geq\sum_{cyc}\frac{3a}{a+b+c}=3$$

as desired. $\square$

2 Mixing Variables

The idea behind mixing variables is to blend the given variables into a more convenient form. Examples include $(x,y)\mapsto(\frac{x+y}{2},\frac{x+y}{2})$, $(x,y)\mapsto(\sqrt{xy},\sqrt{xy})$, $(x,y)\mapsto(x+y,0)$, and $(x,y)\mapsto(xy,1)$. In this sense, mixing variables can be thought of as a variant of smoothing. The requisite computations are often not as clear as they are for Jensen or Karamata, but fortunately they typically do not include Calculus. Moreover, that the computations are less deterministic is a consequence of the number of possibilities.

Show that for reals x,y,z, not all of which are positive,

$$\frac{16}{9}(x^2-x+1)(y^2-y+1)(z^2-z+1)\geq(xyz)^2-xyz+1$$

Solution Suppose without loss of generality that $x\geq y\geq z$. Note that $x^2-x+1\geq|x|^2-|x|+1$. Thus, if $0>x$, exchanging x and y for $x'=-x$ and $y'=-y$ decreases the left hand side while fixing the right hand side. If instead $x\geq 0>y$, simply exhanging y for $y'=-y$ decreases the left hand side while increasing the right hand side. Thus, we may assume $x\geq y\geq 0\geq z$. We now prove a lemma:

Lemma 1 *For any reals $a\geq 0\geq b$, the following inequality holds:*

$$\frac{4}{3}(a^2-a+1)(b^2-b+1)\geq(ab)^2-ab+1$$

Proof. Let $c = -b$. The desired inequality is equivalent to

$$\frac{4}{3}(a^2 - a + 1)(c^2 + c + 1) \geq (ac)^2 + ac + 1,$$

where $a, c \geq 0$, but this equivalent inequality is easily realized as the sum of three inequalities:

$$\begin{aligned}
\frac{4}{3}(a^2 - a + 1)c^2 \geq a^2c^2 \quad &\iff \quad (a-2)^2c^2 \geq 0 \\
\frac{4}{3}(a^2 - a + 1)c \geq ac \quad &\iff \quad (4a^2 - 7a + 4)c \geq 0 \\
\frac{4}{3}(a^2 - a + 1) \geq 1 \quad &\iff \quad (2a-1)^2 \geq 0.
\end{aligned}$$

Now we are done, for by the lemma, we have

$$\frac{16}{9}(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq \frac{4}{3}\left((xz)^2 - xz + 1\right)(y^2 - y + 1) \geq (xyz)^2 - xyz + 1,$$

where $y \geq 0 \geq xz$. □

3 Sum of Squares

Sometimes an inequality is so sharp that none of the usual theorems can be used to solve it. Sharpness clearly greater than Schur, or unusual equality cases are good indicators of this situation. For these problems, carefully chosen squares, or sharp bounds following from pure algebra, or both, may be useful. The reasoning is that weighted sums of squares such that both the squares and the weights become small near equality cases give rise to particularly sharp inequalities.

A good goal for three variable inequalities is to manipulate to the form $\sum f(a, b, c)(a - b)^2 \geq 0$ for some reasonable function f. A variety of conditions then demonstrate the inequality.

(Anh-Cuong) a, b, c are positive reals. Show that

$$a^3 + b^3 + c^3 + 3abc \geq ab\sqrt{2a^2 + 2b^2} + bc\sqrt{2b^2 + 2c^2} + ca\sqrt{2c^2 + 2a^2}$$

Solution. Note that $\frac{3a^2+2ab+3b^2}{4(a+b)} \geq \sqrt{\frac{a^2+b^2}{2}}$, which is a consequence of $(3a^2+2ab+3b^2)^2 - 8(a^2+b^2)(a+b)^2 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 = (a-b)^4 \geq 0$. It therefore suffices to check that

$$a^3 + b^3 + c^3 + 3abc \geq \sum_{cyc} ab \cdot \frac{3a^2 + 2ab + 3b^2}{2(a+b)}$$

or that

$$\sum_{cyc} a(a-b)(a-c) \geq \sum_{cyc} \left[\frac{3a^2 + 2ab + 3b^2}{2(a+b)} - ab(a+b)\right] = \sum_{cyc} \frac{ab(a-b)^2}{2(a+b)}$$

Consider the identity

$$2\sum_{cyc} a^r(a-b)(a-c) = \sum_{cyc} (a^r + b^r - c^r)(a-b)^2.$$

Taking $r = 1$, we need only prove

$$\sum_{cyc} \left(a + b - c - \frac{ab}{a+b}\right)(a-b)^2 \geq 0$$

This is evident, for we may assume without loss of generality that $a \geq b \geq c$ and note that the desired is the sum of the three inequalities

$$\begin{aligned}
\left(a+b-c-\frac{ab}{a+b}\right)(a-b)^2 &\geq 0 \\
\left(c+a-b-\frac{ac}{a+c}\right)\left((a-c)^2-(b-c)^2\right) &\geq 0 \\
\left(b+c-a-\frac{bc}{b+c}\right)(b-c)^2+\left(c+a-b-\frac{ac}{a+c}\right)(b-c)^2 &\geq 0. \ \square
\end{aligned}$$

4 Problems

1. Prove the weighted form of Jensen's inequality from the definition of a convex function: if f be a convex function on an interval I and $x_1, \ldots, x_n \in I$, then for any nonnegative reals $\lambda_1, \ldots, \lambda_n$ with sum 1,
$$\lambda_1 f(x_1)+\cdots+\lambda_n f(x_n) \geq f(\lambda_1 x_1+\cdots+\lambda_n x_n).$$

2. Show that the AM-GM inequality follows from Jensen's inequality: if $a_1, \ldots, a_n$ are positive reals and $\lambda_1, \ldots, \lambda_n$ are nonnegative reals with sum 1, then
$$\lambda_1 a_1+\cdots+\lambda_n a_n \geq a_1^{\lambda_1} \cdots a_n^{\lambda_n}.$$

3. Prove Hölder's inequalty using only AM-GM: if $a_1, \ldots, a_n; b_1, \ldots, b_n; \ldots; z_1, \ldots, z_n$ are sequences of nonnegative reals and $\lambda_a, \ldots, \lambda_z$ are nonnegative reals with sum 1, then
$$(a_1+\cdots+a_n)^{\lambda_a}(b_1+\cdots+b_n)^{\lambda_b} \cdots (z_1+\cdots+z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z}+\cdots+a_n^{\lambda_a} b_n^{\lambda_b} \cdots z_n^{\lambda_z}.$$

4. Use Hölder's inequality to prove the power-mean inequality: if $a_1, \ldots, a_n$ are positive reals and $\lambda_1, \ldots, \lambda_n$ are nonnegative reals with sum 1, then for nonzero reals r and s with $r > s$,
$$\left(\lambda_1 a_1^r+\cdots+\lambda_n a_n^r\right)^{1/r} \geq \left(\lambda_1 a_1^s+\cdots+\lambda_n a_n^s\right)^{1/s}.$$

5. (Titu) Prove that
$$\frac{a}{3b+c}+\frac{b}{3c+d}+\frac{c}{3d+a}+\frac{d}{3a+b} \geq 1$$
for all positive real numbers a, b, c, d.

6. Let $a_1, a_2, \ldots, a_n$ be nonnegative reals with a sum of 1. Prove that
$$a_1 a_2+a_2 a_3+\cdots+a_{n-1} a_n \leq \frac{1}{4}$$

7. (USAMO 97/5) Prove that for all positive reals a, b, c,
$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$$

8. (Japan 97) Let a, b, c be positive real numbers. Prove that
$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

9. (IMO 00) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

10. Let $a_0, a_1, \ldots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

11. (USAMO 01/3) Let a, b, c be nonnegative reals such that

$$a^2 + b^2 + c^2 + abc = 4$$

Prove that

$$0 \leq ab + bc + ca - abc \leq 2$$

12. (Poland 96?) Let a, b, c be real numbers with $a + b + c = 1$ and $a, b, c \geq -3/4$. Prove that

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{9}{10}.$$

13. Show that for all nonnegative reals a, b, c,

$$(a+b+c)^3 \geq 3(ab+bc+ca)\sqrt{3(a^2+b^2+c^2)}$$

14. (Michael Rozenberg) Show that for all positive reals a, b, c,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{3}{2} \cdot \frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

15. (MOP 2006) Show that for nonnegative reals a, b, c,

$$\sqrt{(a^2b+b^2c+c^2a)(a^2c+c^2b+b^2a)} \geq abc + \sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}$$

16. Show that for nonnegative reals a, b, c,

$$2a^6 + 2b^6 + 2c^6 + 16a^3b^3 + 16b^3c^3 + 16c^3a^3 \geq 9a^4(b^2+c^2) + 9b^4(c^2+a^2) + 9c^4(a^2+b^2)$$

17. Show that for all positive reals a, b, c,

$$\sqrt[3]{4a^3+4b^3} + \sqrt[3]{4b^3+4c^3} + \sqrt[3]{4c^3+4a^3} \leq \frac{4a^2}{a+b} + \frac{4b^2}{b+c} + \frac{4c^2}{c+a}$$

18. Show that for all positive numbers $x_1, \ldots, x_n$,

$$\frac{x_1^3}{x_1^2+x_1x_2+x_2^2} + \frac{x_2^3}{x_2^2+x_2x_3+x_3^2} + \cdots + \frac{x_n^3}{x_n^2+x_nx_1+x_1^2} \geq \frac{x_1+\cdots+x_n}{3}$$

19. Prove that for all positive reals a, b, c, d,

$$a^4b + b^4c + c^4d + d^4a \geq abcd(a+b+c+d)$$

20. (USAMO 2003) Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

21. Prove that for all positive reals a, b, c,

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)}\ge 2$$

22. (Bulgaria 97) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\le\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}.$$

23. (Darij Grinberg) Show that for any positive reals a, b, c,

$$\frac{(b+c)^2}{a^2+bc}+\frac{(c+a)^2}{b^2+ca}+\frac{(a+b)^2}{c^2+ab}\ge 6$$

24. (IMO 99) Let $n \ge 2$ be a fixed integer. Find the smallest constant C such that for all nonnegative reals $x_1, \dots, x_n$,

$$\sum_{1\le i<j\le n} x_i x_j (x_i^2+x_j^2)\le C\left(\sum_{1\le i\le n} x_i\right)^4.$$

Determine when equality occurs.

25. (Vietnam 98) Let $x_1, \dots, x_n$ be positive reals such that

$$\frac{1}{x_1+1998}+\frac{1}{x_2+1998}+\cdots+\frac{1}{x_n+1998}=\frac{1}{1998}$$

Prove that

$$\frac{\sqrt[n]{x_1x_2\cdots x_n}}{n-1}\ge 1998$$

26. (Vietnam '96) Let a, b, c, d be four nonnegative real numbers satisfying the condition

$$2(ab+ac+ad+bc+bd+cd)+abc+bcd+acd+bcd=16.$$

Prove that

$$a+b+c+d\ge\frac{2}{3}\left(ab+ac+ad+bc+bd+cd\right),$$

and determine when equality holds.

27. (USAMO 00/6) Let $n \ge 2$ be an integer and $S = \{1, 2, \dots, n\}$. Show that for all nonnegative reals $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$,

$$\sum_{i,j\in S}\min\{a_ia_j, b_ib_j\}\le\sum_{i,j\in S}\min\{a_ib_j, a_jb_i\}$$

28. (George Tsintifas) Prove that for any $a, b, c, d > 0$ we have the inequality

$$(a+b)^3(b+c)^3(c+d)^3(d+a)^3\ge 16a^2b^2c^2d^2(a+b+c+d)^4.$$

29. (MOP 2003) Show that for positive reals $x_1, \ldots, x_n$ satisfying

$$x_1 + \cdots x_n = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

the following inequality holds:

$$\frac{1}{n-1+x_1} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

30. (???) Let n be an integer greater than 2, and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{x_1}{x_2+x_3} + \frac{x_2}{x_3+x_4} + \ldots + \frac{x_{n-1}}{x_n+x_1} + \frac{x_n}{x_1+x_2} > \frac{5n}{12}$$

5 Homework

1. Prove the symmetric polynomials trick by considering the polynomial $P(x) = (x-x_1)\cdots(x-x_m)$ and its derivative.
2. Using the symmetric polynomials trick, prove the Newton inequalities.
3. Prove the Maclaurin inequalities using the symmetric polynomials trick.
4. Suppose that a, b, c, S_a, S_b, S_c are arbitrary reals such that $|S_a|, |S_b|, |S_c|$ satisfy the triangle inequality. Prove that
$$S_a^2(a-b)(a-c) + S_b^2(b-c)(b-a) + S_c^2(c-a)(c-b) \geq 0.$$
5. Let a, b, c be arbitrary reals. Show that the inequality
$$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0$$
holds for each of the following situations:
   - $S_a, S_b, S_c \geq 0$,
   - $a \geq b \geq c$ and $S_b, S_b + S_c, S_b + S_a \geq 0$,
   - $a \geq b \geq c$ and $S_a, S_c, S_a + 2S_b, S_c + 2S_b \geq 0$,
   - $a \geq b \geq c$ and $S_b, S_c \geq 0$, and $a^2 S_b + b^2 S_a \geq 0$.